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New analytical method for the study of natural convection flow of a non-Newtonian fluid

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Abstract

Purpose – The purpose of this paper is to discuss the natural convection flow of an incompressible third grade fluid between two parallel plates. The basic equations governing the flow are reduced to a nonlinear ordinary differential equation.

Design/methodology/approach – The resulting nonlinear ordinary differential equation is solved by multi-step differential transform method (MDTM).

Findings – The obtained solutions in comparison with the numerical solutions (fourth-order Runge-Kutta) admit a remarkable accuracy.

Originality/value – The analysis illustrates the validity and the great potential of the MDTM in solving nonlinear differential equations.

Keywords Differential equations, Convection, Fluids, Flow, Natural convection, Non-Newtonian fluid, Multi-step differential transform method

Paper type Research paper

Nomenclature

E	= Eckert number
i	= unit vector in the direction of x
Pr	= Prandtl number
T	= absolute temperature
U_0	= reference velocity
$v(\eta)$	= velocity function
α	= thermal diffusivity
β	= material coefficient
δ	= dimensionless non-Newtonian coefficient
η	= similarity parameter
μ	= coefficient of viscosity
ν	= kinematic viscosity
$\theta(\eta)$	= temperature function

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1. Introduction

The study of nonlinear ordinary/partial differential equations is quite popular area of research now days. Such equations arise in various physical problems of engineering. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still challenging that needs new methods for exact or approximate solutions. All of nonlinear equations do not have a precise analytic solution and hence numerical methods have largely been used to handle such equations. There are also some analytical techniques for nonlinear equations. Some of the classic analytical methods are the Lyapunov's artificial small parameter method (Lyapunov, 1990), perturbation techniques (He, 1999) and δ-expansion method (Karmishin *et al.*, 1990). In the recent years, many authors mainly had paid attention to study solutions of nonlinear partial differential equations by using various methods. Among these are the Adomian decomposition method (ADM), tanh method, homotopy perturbation method (HPM), sinh-cosh method, homotopy analysis method (HAM) (Rashidi and Dinarvand, 2009; Rashidi *et al.*, 2011; Ellahi, 2009), differential transform method (DTM) (Rashidi and Keimanesh, 2010; Rashidi *et al.*, 2010; Rashidi, 2009) and variational iteration method (VIM) (He, 1997; Rashidi and Shahmohamadi, 2009).

The motivation of this paper is therefore to use the reliable algorithm of the DTM, namely MDTM (Odibat *et al.*, 2010) to construct analytical approximate solutions of the natural convection flow of a third grade between two parallel plates.

Recently considerable attention has been directed towards the study of non-Newtonian fluids because of their practical importance in engineering and industry. The classical Navier-Stokes equations have been proved inadequate to describe and capture the characteristics of complex rheological fluids as well as polymer solutions (Dunn and Rajagopal, 1995). These kinds of fluids are generally known as non-Newtonian fluids. Most of the biological and industrial fluids are non-Newtonian in nature. Few examples of such fluids are blood at low shear rate, tomato ketchup, honey, mud, plastics and polymer solutions. The inadequacy of the classical theories to describe these complex fluids has led to the development of different new models to study non-Newtonian fluids. There are different models which have been proposed to describe the non-Newtonian flow behavior. Among these, the fluids of differential type (Dunn and Rajagopal, 1995; Truesdell and Noll, 2004) have received considerable attention. The third grade fluid considered in this study is a subclass of differential type fluids which can describe the shear thinning/shear thickening effects. Ellahi and Riaz (2010) analyzed the influence of magnetohydrodynamics (MHD) on the pipe flow of a third-grade fluid with variable viscosity. The influence of variable viscosity and viscous dissipation on the non-Newtonian flow in a porous medium employing modified Darcy's law is discussed by Hayat *et al.* (2007) and Ellahi and Afzal (2009). Okoya considered the flow of a third grade fluid (with exponential viscosity) between the parallel plates under the action of externally imposed pressure gradient (Okoya, 2011). Yürüsoy *et al.* (2008) considered the steady-state flow of a third grade fluid between concentric circular cylinders. They examined entropy generation due to fluid friction and heat transfer in the annular pipe. Pakdemirli and Yilbas (2006) considered flow of third grade fluid in a pipe. Ayub *et al.* (2003) considered the flow of third grade fluid past a porous plate. The thermal transition in reactive flow of a third-grade fluid with viscous heating and chemical reaction between two horizontal flat plates is examined by Okoya (2008). Akyildiz *et al.* 2004 developed exact solutions for nonlinear differential equations of third grade fluid. Thermal stability of reactive third grade fluid in a cylindrical pipe is analyzed by Makinde (2007). Sahoo (2009) examined Hiemenz flow of a third grade fluid in the presence of heat transfer. Fundamental Stokes first problem for third grade fluid in a porous space is studied by Hayat *et al.* (2008). The authors introduced the modified Darcy's law for third grade fluid in this work. Abelman *et al.* (2008) discussed the rotating flow of third grade fluid in a porous space. Exact solution for flow of third grade fluid over a porous wall is established by Hayat *et al.* (2003).

2. Basic concepts of the DTM

Transformation of the *k*th derivative of a function in one variable is as follows (Abdel-Halim Hassan, 2008):

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(t)}{dt^k}\right]_{t=t_0}, \tag{1}$$

and the inverse transformation is defined by:

$$f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^k. \tag{2}$$

From equations (1) and (2), we get:

$$f(t) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} \frac{d^k f(t)}{dt^k}\bigg|_{t=t_0}, \tag{3}$$

which implies that the concept of DTM is resulting from Taylor series expansion, but the method does not calculate the derivatives representatively. However, relative derivatives are calculated by an iterative method which is described by the transformed equations of the original function. For implementation purposes, the function $f(t)$ is expressed by a finite series and equation (2) can be written as:

$$f(t) \cong \sum_{k=0}^{i} F(k)(t - t_0)^k, \tag{4}$$

where $F(k)$ is the differential transform of $f(t)$.

The following theorems that can be deduced from equations (21) and (22) are given below:

Theorem 1. If $f(t) = u(t) \pm v(t)$, then $F(k) = U(k) \pm V(k)$.

Theorem 2. If $f(t) = \lambda u(t)$, then $F(k) = \lambda U(k)$, where λ is a constant.

Theorem 3. If $f(t) = d^n u(t)/dt^n$, then $F(k) = ((k+n)!/k!)U(k+n)$.

Theorem 4. If $f(t) = u(t)v(t)$, then $F(k) = \sum_{r=0}^{k} U(r)V(k-r)$.

Theorem 5. If $f(t) = (du(t)/dt)(du(t)/dt)$, then
$F(k) = \sum_{r=0}^{k}(r+1)(k-r+1)U(r+1)U(k-r+1)$.

Theorem 6. If $f(t) = (du(t)/dt)(dv(t)/dt)$, then
$F(k) = \sum_{r=0}^{k}(r+1)(k-r+1)U(r+1)V(k-r+1)$.

Theorem 7. If $f(t) = u(t)(dv(t)/dt)$, then $F(k) = \sum_{r=0}^{k}(k-r+1)U(r)V(k-r+1)$.

3. Basic concepts of the MDTM

When the DTM is used for solving differential equations with the boundary conditions at infinity or problems that have highly nonlinear behavior, the obtained results were incorrect (when the boundary-layer variable goes to infinity, the obtained series solutions are divergent). Besides that, power series are not useful for large values of independent variable.

To overcome the shortcoming, the MDTM is presented in this section that has developed for the analytical solution of differential equations. For this purpose, the following nonlinear initial value problem is considered:

$$u(t, f, f', \ldots, f^{(p)}) = 0, \tag{5}$$

subject to the initial conditions $f^{(k)}(0) = c_k$, for $k = 0, 1, \ldots, p-1$.

Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (5). In actual applications of the DTM, the approximate solution of the initial value problem (5) can be expressed by the finite series:

$$f(t) = \sum_{n=0}^{N} a_n (t - t_0)^n \quad t \in [0, T]. \tag{6}$$

The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[0, T]$ is divided into M subintervals $[t_{m-1}, t_m], m = 1, 2, \ldots, M$ of equal step size $h = T/M$ by using the nodes $t_m = mh$. The main idea of the MDTM is as follows. First, we apply the DTM to equation (5) over the interval $[0, t_1]$ and obtain the following approximate solution:

$$f_1(t) = \sum_{n=0}^{K} a_{1n} (t - t_0)^n \quad t \in [0, t_1], \tag{7}$$

with the initial conditions $f_1^{(k)}(0) = c_k$. For $m \geq 2$ and at each subinterval $[t_{m-1}, t_m]$ we will use the initial conditions $f_m^{(k)}(t_{m-1}) = f_{m-1}^{(k)}(t_{m-1})$ and apply the DTM to equation (5) over the interval $[t_{m-1}, t_m]$, where t_0 in equation (1) is replaced by t_{m-1}. The process is repeated and generates a sequence of approximate solutions $f_m(t), m = 1, 2, \ldots, M$, for the solution $f(t)$:

$$f_m(t) = \sum_{n=0}^{K} a_{mn} (t - t_{m-1})^n, \quad t \in [t_m, t_{m+1}], \tag{8}$$

where $N = K \cdot M$. In fact, the MDTM assumes the following solution:

$$f(t) = \begin{cases} f_1(t), & t \in [0, t_1] \\ f_2(t), & t \in [t_1, t_2] \\ \vdots & \\ f_M(t), & t \in [t_{M-1}, t_M]. \end{cases} \tag{9}$$

The new algorithm, MDTM, is simple for computational performance for all values of h. It is easily observed that $h = T$, if the step size then the MDTM reduces to the classical DTM. As we will see in the next section, the main advantage of the new algorithm is that the obtained series solution converges for wide time regions and can approximate non-chaotic or chaotic solutions.

4. Mathematical formulation

The structure of the fully developed steady-state flow of an incompressible liquid of grade 3 confined between two parallel plates is studied in this paper. The velocity for unidirectional flow is:

$$\vec{V} = u(y)\vec{i}, \tag{10}$$

in which $\vec{V}$ is the velocity, u and $\vec{i}$ are velocity and unit vector parallel to the x-axis. The equations motion and energy give (Rajagopal and Na, 1985):

$$\frac{d^2u}{dy^2}\left(\mu + 6\beta\left(\frac{du}{dy}\right)^2\right) + \frac{T - T_0}{\beta T_0} = 0, \tag{11}$$

$$k\frac{d^2T}{dy^2} + \left(\frac{du}{dy}\right)^2\left(\mu + 2\beta\left(\frac{du}{dy}\right)^2\right) = 0, \tag{12}$$

where continuity equations is identically satisfied and the consumption of the combustible material and the effect of radiant heating are neglected. The dimensionless variables are defined as:

$$\eta = \frac{y}{y_0}, \quad v = \frac{u}{U_0}, \quad \beta = \frac{RT_0}{E}, \quad E = \frac{U_0^2}{c(T_1 - T_2)}, \quad Pr = \frac{\nu}{\alpha}, \quad \theta = \frac{T - T_0}{\beta T_0}, \quad \delta = \frac{\beta U_0^2}{\mu y_0^2}, \tag{13}$$

where c is the specific heat of the fluid. The non-dimensional form of equations (11) and (12) are:

$$\frac{d^2v}{d\eta^2}\left(1 + 6\delta\left(\frac{dv}{d\eta}\right)^2\right) + \theta = 0, \tag{14}$$

$$\frac{d^2\theta}{d\eta^2} + E \cdot Pr\left(\frac{dv}{d\eta}\right)^2\left(1 + 2\delta\left(\frac{dv}{d\eta}\right)^2\right) = 0. \tag{15}$$

With the following boundary conditions:

$$v(-1) = 0, \quad \theta(-1) = \frac{1}{2}, \tag{16}$$

$$v(+1) = 0, \quad \theta(+1) = -\frac{1}{2}. \tag{17}$$

The physical model of the problem is shown in Figure 1. It consists of two flat plates that can be positioned vertically. A non-Newtonian fluid is between two flat plates distant $2b$ apart. The walls at $x = +b$ and $x = -b$ are kept at constant temperatures T_2 and T_1, respectively, where $T_1 > T_2$. This difference in temperature causes the fluid near the wall at $x = -b$ to rise and the fluid near the wall at $x = +b$ to fall.

5. Analytical solutions by MDTM

Employing the MDTM to equations (11) and (12) gives the following recursive relation in each sub-domain $(t_i, t_{i+1}), i = 0, 1, \ldots, N-1$:

$$\begin{aligned}(k+1)(k+2)V(k+2) + 6\delta \sum_{r_2=0}^{k}\sum_{r_1=0}^{r_2} V(r_1+1)V(r_2-r_1+1)V(k-r_2+2) \\ \times (r_1+1)(r_2-r_1+1)(k-r_2+1)(k-r_2+2) + \Theta(k),\end{aligned} \tag{18}$$

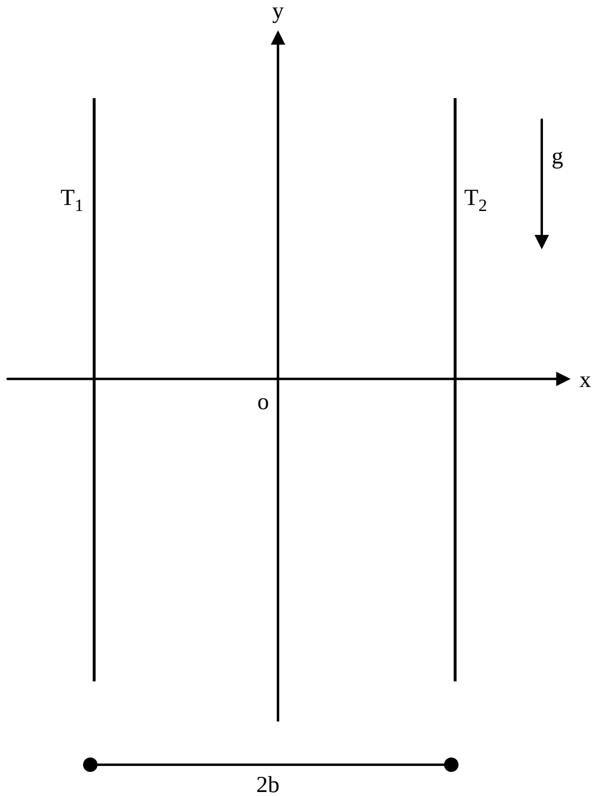

Figure 1.
Schematic diagram of the problem under consideration

$$
\begin{aligned}
&(k+1)(k+2)\Theta(k+2)+E\cdot Pr\sum_{r=0}^{k}V(r+1)V(k-r+1)(r+1)(k-r+1)\\
&+2\delta E\cdot Pr\sum_{r_3=0}^{k}\sum_{r_2=0}^{r_3}\sum_{r_1=0}^{r_2}V(r_1+1)V(r_2-r_1+1)V(r_3-r_2+1)V(k-r_3+1)\\
&\times(r_1+1)(r_2-r_1+1)(r_3-r_2+1)(k-r_3+1),
\end{aligned}
\tag{19}
$$

where $V(k)$ and $\Theta(k)$ are the differential transforms of $v(\eta)$ and $\theta(\eta)$.

The differential transform of the boundary conditions (16) and (17) are as follows:

$$V(0)=0,\quad \Theta(0)=\frac{1}{2}, \tag{20}$$

$$\sum_{k=0}^{i}V(k)2^k=0,\quad \sum_{k=0}^{i}\Theta(k)2^k=-\frac{1}{2}, \tag{21}$$

We can consider the following boundary conditions (equations (16) and (17)):

$$v(-1)=0,\quad \theta(-1)=\frac{1}{2}, \tag{22}$$

$$v'(-1)=\alpha,\quad \theta'(-1)=\beta. \tag{23}$$

The differential transform of the above conditions are given by:

$$V(0)=0,\quad \Theta(0)=\frac{1}{2}, \tag{24}$$

$$V(1)=\alpha,\quad \Theta(1)=\beta. \tag{25}$$

Moreover, substituting equations (24) and (25) into equations (18) and (19) and by recursive method we can calculating other values of $V(k)$ and $\Theta(k)$. Hence, substituting all $V(k)$ and $\Theta(k)$, into equation (4), we have series solutions. By using boundary condition $v(1)=0$ and $\theta(1)=-(1/2)$, we can obtain α, β.

For analytical solution, the convergence analysis is performed and in equation (4), the i value is selected equal to 20. We set interval equal to 0.1.

6. Results and discussion

Our main concern is to determine solutions of velocity and temperature profiles, $v(\eta), \theta(\eta)$, by the DTM, MDTM and numerical method using the fourth-order Runge-Kutta. These quantities describe the flow behavior.

Figures 1 and 2 show the accuracy of the MDTM solution in comparison to the DTM and numerical solution when $Pr=1, \delta=1$ and various values of E, for $v(\eta)$ and $\theta(\eta)$, respectively. It is observed that $v(\eta)$ and $\theta(\eta)$, are increased when Eckert number E, increases. The effects of Prandtl number on $v(\eta)$ and $\theta(\eta)$ are shown in Figures 3 and 4. It is clear that as Prandtl number increases, the $v(\eta)$ and $\theta(\eta)$ increase. Variation of $v(\eta)$ and $\theta(\eta)$, with respect to δ is presented in Figures 5 and 6.

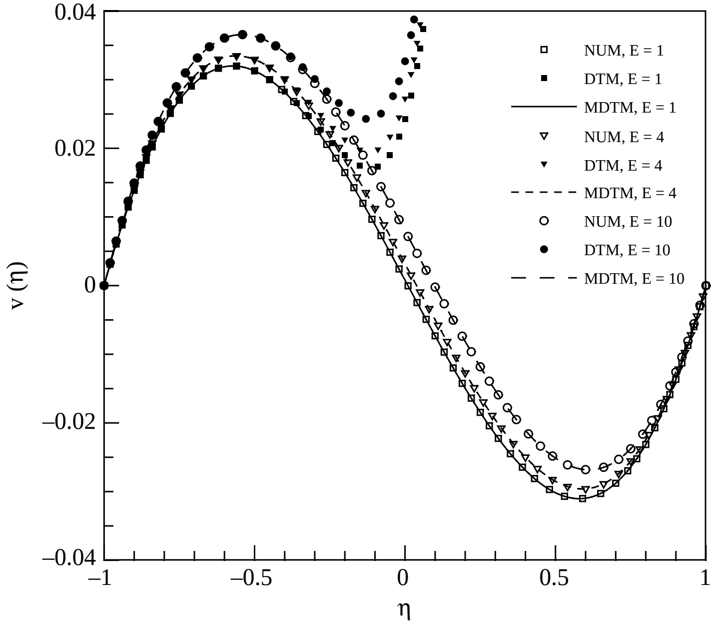

Figure 2.
Variation of velocity profile v(η) with respect to E, when $\mathrm{Pr} = 1$ and $\delta = 1$

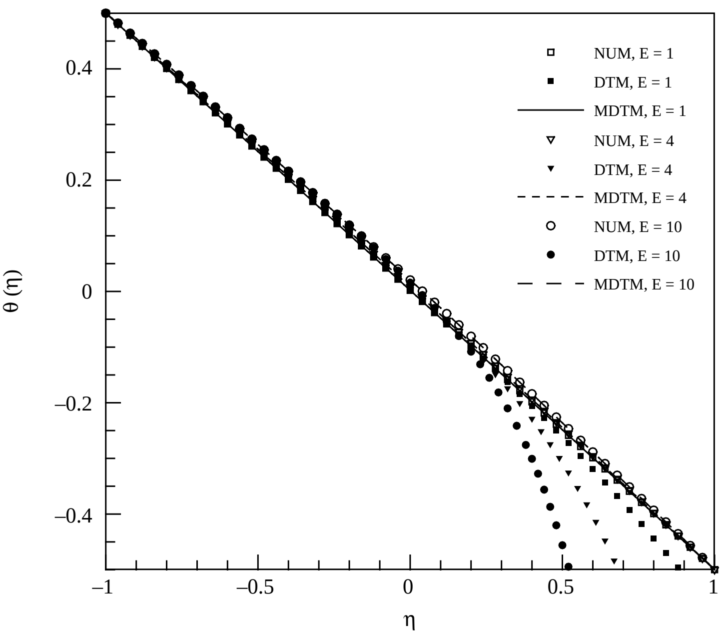

Figure 3.
Variation of temperature profile $\theta(\eta)$ with respect to E, when $\mathrm{Pr} = 1$ and $\delta = 1$

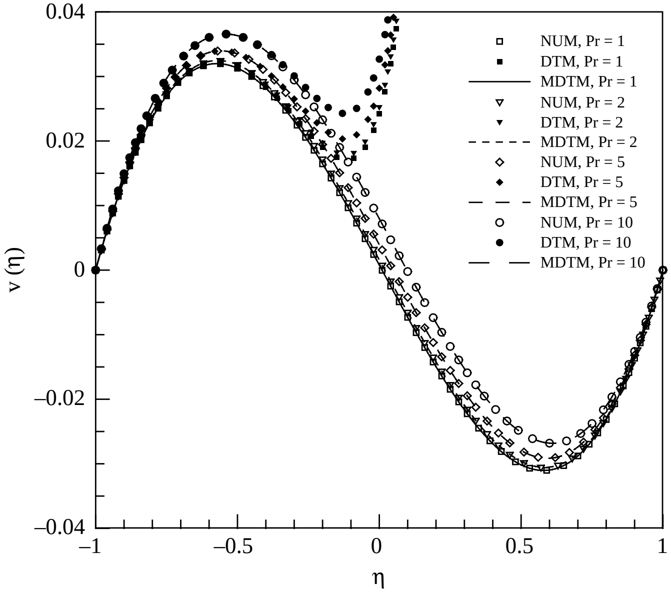

Figure 4.
Variation of velocity profile v(η) with respect to *Pr*, when Pr $= 1$ and $\delta = 1$

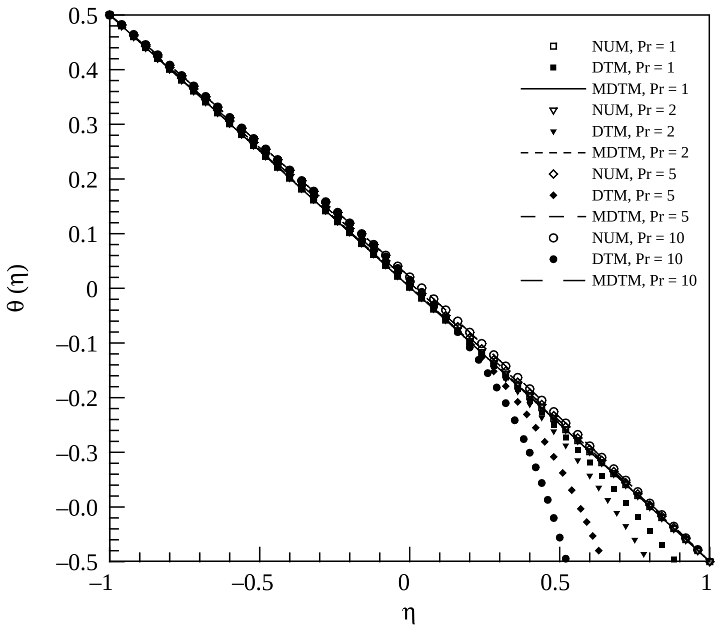

Figure 5.
Variation of temperature profile $\theta(\eta)$ with respect to *Pr*, when Pr $= 1$ and $\delta = 1$

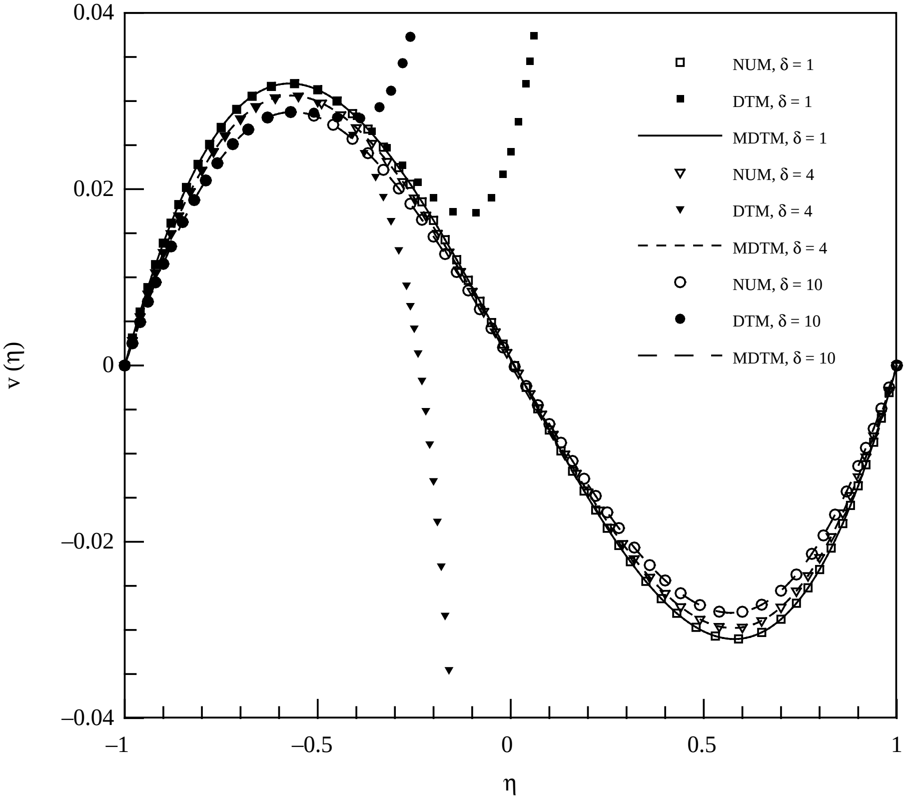

Figure 6.
Variation of velocity profile $v(\eta)$ with respect to δ, when $E = 1$ and $\Pr = 1$

It is observed that increasing parameter δ causes a decrease in $v(\eta)$ but it has not significant effect on $\theta(\eta)$.

In order to verify the efficiency of the proposed method in comparison with the DTM and numerical solution, a comparison is presented in Tables I and II for different values of parameter δ.

7. Conclusion

In the present study, a reliable algorithm based on the DTM is presented to solve some nonlinear equations. The present method reduces the computational difficulties of the other methods (same as the HAM, VIM, ADM and HPM).

The results for $v(\eta)$ and $\theta(\eta)$, in Tables I and II illustrated the validity and accuracy of this procedure. Note that the MDTM is easier to calculate than the other methods because in the MDTM we have an iterative procedure which does not need to solve any differential or integral equations. In the other methods we must in each iterate solve differential equations or integrate the equations (Figure 7).

Table I.
Comparison of obtained results for $v(\eta)$, when $Pr = 1$, $E = 1$ and various values of parameter δ

η	$v(\eta)$ $\delta = 1$ NUM	$\delta = 1$ DTM	$\delta = 1$ MDTM	$\delta = 4$ NUM	$\delta = 4$ DTM	$\delta = 4$ MDTM	$\delta = 10$ NUM	$\delta = 10$ DTM	$\delta = 10$ MDTM
−1	0	0	0	0	0	0	0	0	0
−0.9	0.0138792	0.0138792	0.0138792	0.0128104	0.0128104	0.0128104	0.0115298	0.0115298	0.0115298
−0.8	0.0236092	0.0236092	0.0236092	0.0221529	0.0221529	0.0221529	0.020283	0.020283	0.020283
−0.7	0.0294797	0.0294798	0.0294798	0.0279906	0.0279901	0.0279906	0.0260053	0.0260066	0.0260053
−0.6	0.0318811	0.0318829	0.0318811	0.0304751	0.0304607	0.0304751	0.0285754	0.0286113	0.0285754
−0.5	0.0312751	0.0312949	0.0312751	0.0299666	0.0297833	0.0299666	0.0281831	0.0285963	0.0281831
−0.4	0.0281619	0.028296	0.0281619	0.0269687	0.0255922	0.0269687	0.0253301	0.0279773	0.0253301
−0.3	0.0230536	0.0237117	0.0230537	0.0220305	0.0148638	0.0220305	0.0206243	0.0320515	0.0206243
−0.2	0.0164582	0.0190119	0.0164583	0.0156827	−0.01307	0.0156827	0.0146241	0.0524216	0.014624
−0.1	0.0088734	0.0171707	0.00887341	0.00841798	−0.086871	0.00841799	0.00780665	0.111775	0.00780663
0	0.00078687	0.0242641	0.000786885	0.000695376	−0.272639	0.00069539	0.000588839	0.250956	0.000588807
0.1	−0.0073192	0.052173	−0.00731922	−0.00704414	−0.707836	−0.0070441	−0.00664268	0.538868	−0.00664272
0.2	−0.0149624	0.12286	−0.0149624	−0.0143598	−1.65719	−0.0143598	−0.0135018	1.0857	−0.0135018
0.3	−0.0216537	0.274795	−0.0216537	−0.0207928	−3.60063	−0.0207927	−0.0195735	2.05982	−0.0195736
0.4	−0.0268935	0.572247	−0.0268935	−0.0258508	−7.36648	−0.0258508	−0.0243835	3.70858	−0.0243835
0.5	−0.0301707	1.11826	−0.0301707	−0.0290021	−14.3268	−0.0290021	−0.0273753	6.38285	−0.0273753
0.6	−0.0309691	2.07232	−0.030969	−0.0296932	−26.676	−0.0296931	−0.0279371	10.5648	−0.0279372
0.7	−0.0287827	3.67386	−0.0287826	−0.0274108	−47.8193	−0.0274108	−0.0255506	16.8981	−0.0255506
0.8	−0.0231422	6.27293	−0.0231422	−0.0217797	−82.9025	−0.0217797	−0.0200027	26.2187	−0.0200027
0.9	−0.013648	10.3695	−0.013648	−0.0126338	−139.523	−0.0126338	−0.0114017	39.5837	−0.0114018
1	0	16.6631	0	0	−228.664	0	0	58.2962	0

Table II. Comparison of obtained results for $\theta(\eta)$, when $Pr = 1$, $E = 1$ and various values of parameter δ

	$\theta(\eta)$								
	$\delta = 1$			$\delta = 4$			$\delta = 10$		
η	NUM	DTM	MDTM	NUM	DTM	MDTM	NUM	DTM	MDTM
−1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
−0.9	0.450438	0.450438	0.450438	0.450418	0.450418	0.450418	0.450391	0.450391	0.450391
−0.8	0.400729	0.400729	0.400729	0.400698	0.400698	0.400698	0.400654	0.400654	0.400654
−0.7	0.350957	0.350957	0.350957	0.350915	0.350915	0.350915	0.350857	0.350857	0.350857
−0.6	0.301166	0.301166	0.301166	0.301113	0.301113	0.301113	0.30104	0.30104	0.30104
−0.5	0.251373	0.251373	0.251373	0.251308	0.25131	0.251308	0.25122	0.25122	0.25122
−0.4	0.201575	0.201572	0.201575	0.201499	0.201511	0.201499	0.201397	0.201396	0.201397
−0.3	0.151759	0.151747	0.151759	0.151673	0.151728	0.151673	0.151558	0.151571	0.151558
−0.2	0.101908	0.101862	0.101908	0.101813	0.102014	0.101813	0.101687	0.101795	0.101687
−0.1	0.052005	0.0518618	0.0520049	0.0519049	0.0525036	0.0519049	0.0517715	0.0522697	0.0517715
0.0	0.00203924	0.00165559	0.00203924	0.00193719	0.00347855	0.00193719	0.00180116	0.00353796	0.00180116
0.1	−0.0479935	−0.0489074	−0.0479935	−0.0480938	−0.0445624	−0.0480938	−0.0482274	−0.0431712	−0.0482274
0.2	−0.0980898	−0.100063	−0.0980898	−0.0981846	−0.0908231	−0.0981846	−0.0983111	−0.0853659	−0.0983111
0.3	−0.148239	−0.152156	−0.148239	−0.148325	−0.134135	−0.148325	−0.148441	−0.118367	−0.148441
0.4	−0.198424	−0.20565	−0.198424	−0.198501	−0.172907	−0.198501	−0.198603	−0.133891	−0.198603
0.5	−0.248629	−0.261101	−0.248629	−0.248694	−0.205127	−0.248694	−0.248782	−0.117935	−0.248782
0.6	−0.298839	−0.319077	−0.298839	−0.298893	−0.228454	−0.298893	−0.298965	−0.0477411	−0.298965
0.7	−0.349052	−0.379967	−0.349052	−0.349093	−0.240465	−0.349093	−0.34915	0.112497	−0.34915
0.8	−0.399281	−0.443657	−0.399281	−0.399311	−0.239113	−0.399311	−0.399354	0.417533	−0.399354
0.9	−0.44957	−0.508989	−0.44957	−0.449589	−0.223504	−0.449589	−0.449615	0.949037	−0.449615
1.0	−0.5	−0.572923	−0.5	−0.5	−0.195094	−0.5	−0.5	1.8262	−0.5

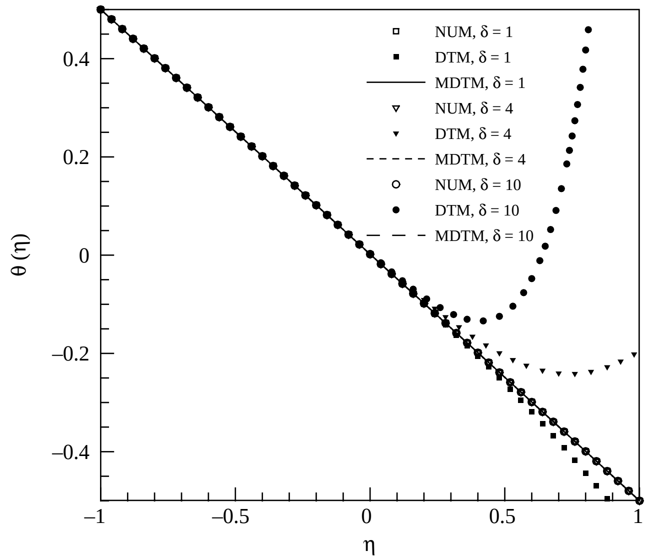

Figure 7.
Variation of temperature profile $\theta(\eta)$ with respect to δ, when $E=1$ and $\Pr=1$.

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