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Stochastic sliding mode control of active vehicle suspension with mismatched uncertainty and multiplicative perturbations

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Abstract

The purpose of this paper is to investigate the stochastic sliding mode controller design for uncertain model of vehicle suspension. The Itô stochastic model of quarter-car is considered applying both parametric stochastic perturbations and mismatched uncertainty of road disturbance. To tackle with uncertainties of model a non-semi-martingale stochastic sliding dynamic is obtained employing a proportional-integral switching surface. By means of linear matrix inequalities (LMIs) and stochastic extension of Lyapunov method, a sufficient condition is derived to guarantee the mean-square stability of the stochastic dynamics in the specified switching surface for all admissible mismatched uncertainties. Furthermore, the synthesized sliding mode controller guarantees the reachability of the determined sliding surface. A simulation study is performed to evaluate the effectiveness of stochastic sliding mode control approach.

KEYWORDS

itô lemma, itô stochastic model, sliding mode control, stochastic uncertainty, suspension system

1 | INTRODUCTION

The vehicle suspension system plays a crucial role in shaping the dynamics of the vehicle and providing an effective balance between ride and handling characteristics. The chief objective of suspension is to isolate the chassis from road roughness in addition to supporting vehicle weight and keeping tire contact with road surface [1]. Consequently, suspension system guarantees passengers stability and comfort by absorbing the road-induced vibrations. Intrinsically, passive suspension can only perform well in a restricted range of operation and therefore the development of control strategies for active/semi-active suspension system enjoys a growing attraction in recent years [2].

Suspension model is subjected to random uncertainties, which arise from external excitation to the system and from the system parameters. External disturbances mainly stem from road irregularities which are induced to system in different channel from the control input, and therefore, lead to mismatched condition [3]. Thus, combining suspension system parameters with established control strategies, it is quite straightforward to synthesize a control system for an active vehicle suspension.

But, in reality, there are a number of practical problems associated with these techniques. Firstly, there is the question of defining suspension model parameters. For a real suspension system, the detailed physical data are often not readily available. Furthermore, vehicle designers usually have access to a large database of suspension components. Therefore, it may be necessary to include uncertainties in approximate model parameters [4]. Secondly, there is the problem of complexities within the suspension system which cannot be easily modeled, such as interactions with braking and steering systems, inaccurate modulus, internal material damping and geometrical variations, etc [5]. Finally, the damping and spring characteristics of the vehicle suspension are inherently nonlinear and are related to vertical velocity and vertical displacement of the wheels (which follow vertical displacements of the road). On the other hand, the road surface itself has a completely stochastic nature and can be satisfactorily modeled as a stationary random process [6]. Thus, it can be concluded that these stochastic uncertainties will be appeared in spring and damping parameters of suspension vehicle [7,8].

There are many research works emphasize on the importance of dealing with the parametric uncertainties of suspension, because of the fact that parameters may vary considerably in vibrational systems [9,10]. To tackle with such uncertainties, robust control [2] and adaptive control [11–13] strategies have been applied to system. Despite the fact that, the parametric perturbations in vibrational systems have random nature [8], in almost all these study works parametric uncertainties are regarded as deterministic bounded signals. While, considering parameters uncertainty as norm-bounded signal implies high risk of conservatism to system, stochastic strategy leads to improvement of system performance.

Extending the model of systems to Itô stochastic differential equations allows us to deal with a wide range of parametric uncertainties in terms of Brownian motion random process. Itô stochastic models have been one of the most practical stochastic models in applications like, economics, finance, flight control, vibrational systems and biology [8,14,15]. There are plenty of studies trying to tackle with stabilizing and control of Itô models through designing an idiosyncratic control strategy. See, for instance [16,17], for stability results, and [18–20] for some recent control strategy.

On the other hand, sliding mode control (SMC), which is robust to parameter variations and external disturbances, has found a wide range of applications in control of uncertain systems [21–24]. Due to its remarkable features such as fast response, appropriate transient response and order reduction, SMC has received a great deal of attention in active suspension control. To tackle with mismatched uncertainty, [3] considered road profile as an unknown nonlinear function and proposed a class of proportional-integral SMC for active suspension. [25] proposed a Fuzzy Sliding Mode Control to estimate state variables and model uncertainties. In order to reduce the acceleration of the sprung mass, a novel disturbance observer based SMC is introduced in [26]. Regardless of carrying out SMC design for vehicle suspension, it can be figured out that parametric stochastic perturbations in suspension system characteristics have not fully taken into account.

The objective of this paper is to improve the ride comfort of passengers using active suspension. Thus, a new Itô-based SMC control scheme is presented, where the combination of Itô stochastic equations of model and robust control technique is utilized to improve the performance of active suspension subject to simultaneous mismatched uncertainty and multiplicative perturbations. The main contributions of this paper can be summarized as follows: (i) constructing an Itô uncertain model of vehicle by considering parametric uncertainties as three independent identically distributed Brownian motion stochastic processes. (ii) Developing an Itô process as the sliding surface which enables us to obtain sufficient stability conditions at sliding mode. Applying stochastic extension of Lyapunov stability the asymptotic stability of the closed-loop dynamics on the sliding hyper-surface is given in terms of LMIs. Furthermore, utilizing infinitesimal operator and Itô lemma, the reachability of the sliding hyper-surface is guaranteed almost surely. (iii) Implementation of proposed Itô-based sliding mode controller to improve suspension ride comfort in presence of uncertainties. Inherently, proposed stochastic sliding mode controller can cope with both parametric perturbations and mismatched uncertainty.

The reminder of this paper is organized as follows. The Itô stochastic model of vehicle suspension system is elicited in Section 2. The problem of stochastic sliding mode controller is formulated in Section 3 and the stability of sliding dynamics and closed loop system is investigated in this section. In order to investigate the effectiveness of stochastic sliding mode method, a numerical simulation is performed in Section 4 and some concluding remarks are given in Section 5.

Notation. $\Phi(\Omega, F, P)$ is a probability space with Ω the sample space, F, the sigma-algebra of subsets of the sample space, and P, the probability measure. $E(.)$ Denotes the expectation operator with respect to probability measure P. For a real symmetric matrix, $M > 0$ means that M is positive definite. $\lambda(M)$ Is defined as eigenvalues of matrix M and $\|g(t)\|$ represents the Euclidian norm of $g(t)$. Infinitesimal generator $\mathscr{L}(.)$ is Itô differential operator which is defined as follows:

$$\mathscr{L}(.) = \frac{\partial(.)}{\partial(x)} f(x,t) + \frac{1}{2} tr\left[g(x,t)g^T(x,t)\frac{\partial^2(.)}{\partial(x)^2}\right], \quad (1)$$

where the Itô stochastic model is as follows:

$$\begin{cases} d\vec{X} = f(X(t), u(t), t)dt + g(X(t), u(t), t)dw \\ \vec{X}(0) = \vec{X}_0 \end{cases} \quad (2)$$

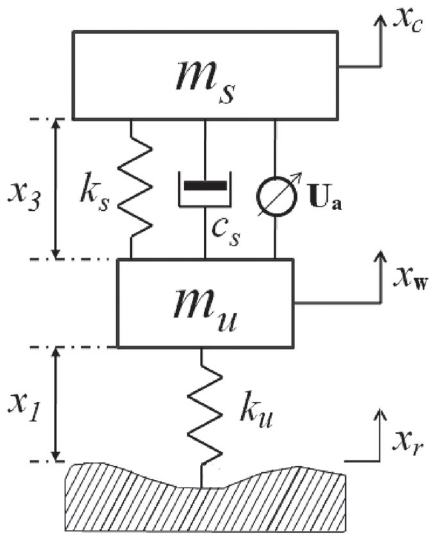

FIGURE 1 Quarter-car model

2 | SYSTEM MODEL

To study the vibrational behavior of vehicle, a quadratic car model consists of rigid bodies and dynamic elements of spring and damping is used as shown in Figure 1. Here, we extend this well-known model to Itô stochastic model of vehicle by considering the parametric stochastic perturbations. By applying Newton's laws to this model, dynamics of system can be described by following differential equations:

$$\begin{aligned}\ddot{x}_c &= \frac{1}{m_s}\left[k_s(x_w - x_c) + c_s(\dot{x}_w - \dot{x}_c) + U_a\right] \\ \ddot{x}_w &= \frac{1}{m_u}\left[k_s(x_c - x_w) - c_s(\dot{x}_w - \dot{x}_c) - U_a \right. \\ &\quad \left. + k_u(x_r - x_w)\right],\end{aligned} \tag{3}$$

where m_s and m_u are the masses of car body and wheel, respectively, x_c And x_w are the displacement of car body and wheel respectively, k_s and k_u are the stiffness coefficients of sprung and unsprang bodies, c_s is the damper coefficient, x_r is the road disturbance and U_a is control input which is applied to vehicle by means of an actuator.

Mechanical systems are subject to uncertainties from external loadings such as wind loading, road roughness and aerodynamic forces and from randomness of material parameters such as inaccurate modulus, internal material damping and geometrical variations due to variabilities of manufacturing processes. By having sufficient amount of data to form a sample space, these uncertainties can be modeled as random variables or stochastic processes by means of statistical inference which are applicable to vibrational systems such as suspension model [8,27]. Here, stochastic perturbations are applied to model (3) as three independent identically distributed Gaussian white noises namely v_1, v_2 and v_3 which present parametric uncertainties of suspension damping and spring characteristics[8]. Thus, the perturbed model of suspension will be as follows:

$$\begin{aligned}\ddot{x}_c &= \frac{1}{m_s}\left[k_s(1+v_1)(x_w - x_c) + c_s(1+v_2)(\dot{x}_w - \dot{x}_c) + U_a\right] \\ \ddot{x}_w &= \frac{1}{m_u}\left[k_s(1+v_1)(x_c - x_w) - c_s(1+v_2)(\dot{x}_w - \dot{x}_c) \right. \\ &\quad \left. - U_a + k_u(1+v_3)(x_r - x_w)\right],\end{aligned} \tag{4}$$

Two equations described in (4) can be rewritten in the following state space form:

$$\begin{aligned}d\vec{X} = &\left\{A\vec{X} + BU_a + Cr\right\}dt + \left\{D\vec{X}\right\}dw_1 + \\ &\left\{E\vec{X}\right\}dw_2 + \left\{F\vec{X}\right\}dw_3,\end{aligned} \tag{5}$$

where the state variables $\vec{X}$ are selected as $\vec{X} = \{x_c - x_w, \dot{x}_c, x_w - x_r, \dot{x}_w\}'$ and:

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{c_s}{m_s} & 0 & \frac{c_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{c_s}{m_u} & -\frac{k_u}{m_u} & -\frac{c_s}{m_u} \end{bmatrix}; B = \begin{bmatrix} 0 \\ \frac{1}{m_s} \\ 0 \\ -\frac{1}{m_u} \end{bmatrix};$$

$$C = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}; D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{k_s}{m_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_s}{m_u} & 0 & 0 & 0 \end{bmatrix};$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{c_s}{m_s} & 0 & \frac{c_s}{m_s} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{c_s}{m_u} & 0 & -\frac{c_s}{m_u} \end{bmatrix}; F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{k_u}{m_u} & 0 \end{bmatrix}$$

And $\vec{w}$ is three-dimensional Brownian motion defined on the probability space $\Phi(\Omega, F, P)$ which is defined as follows:

$$\vec{w} = \begin{bmatrix} dw_1 & dw_2 & dw_3 \end{bmatrix}^T = \begin{bmatrix} v_1 dt & v_2 dt & v_3 dt \end{bmatrix}^T \tag{6}$$

The Equation 6 represents one of the principal characteristics of Brownian motion process which implies that the derivative of a Brownian motion is a white Gaussian noise. In fact, by applying this equation and defining state variables as $\vec{X}$, the Equation 5 will be elicited. In addition, Equation 5 shows that the road disturbance is not in phase with the actuator input, therefore the system suffers from mismatched condition. Hence, in order to deal with mismatched condition, Equation 5 can be written as follows:

$$d\vec{X} = \left\{ A\vec{X} + BU_a + f(t) \right\} dt + \left[D\vec{X} \;\; E\vec{X} \;\; F\vec{X} \right] d\vec{w}, \quad (7)$$

where we use nonlinear uncertain function $f(t)$ to represent the uncertainties with the mismatched condition, i.e. $rank\,[B\,|\,f(t)\,] \neq rank\,[B]$. It is assumed that there exists a known positive constant such that $\|f(t)\| \leq \beta$. Equation 7 demonstrates an Itô stochastic model of suspension system established on the probability space Φ. This model contains the multiplicative stochastic perturbations in system parameters as well as mismatched uncertainty.

Assumption 1. The input matrix B has full column rank and the pair (A, B) is controllable.

3 | STOCHASTIC SLIDING MODE CONTROLLER

First of all, some concepts about stability of stochastic systems are addressed which are defined in [28,29].

Definition 1. The stochastic system described by Itô stochastic Equations 7 with $U = 0$ is said to be mean-square stochastically stable if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\underset{t_0 \leq t < \infty}{\mathrm{SUP}} \; E|X(t)|^2 \leq \varepsilon \;,\; for\, all\, X(t_0) = X_0 \;,\; |X(t)| \leq \delta$$

In addition, it's said to be mean-square asymptotically stable if there exists a $\varepsilon > 0$, such that:

$$\lim_{t \to \infty} E|X(t)|^2 = 0 \quad , \quad X(t_0) = X_0$$

Corollary 1. *Assume that there exists a function* $V \in C^{1,2}(R^n \times [t_0, \infty)\,; R_+)$ *and constants* $c_1 > 0$ *and* $c_2 > 0$ *such that for all* $X \neq X_0$ *and* $t \geq t_0$,

$$(i)\; c_1\,|X| \leq V(X, t) \leq c_2\,|X|$$

$$(ii)\; \mathscr{L}\,(V(X, t)) \leq 0$$

Then, the uncertain system described by Itô stochastic Equations 7 is mean-square asymptotically stable.

Definition 2. The stochastic system described by Itô stochastic Equations 7 is said to be almost surely exponentially stable if for all $X_0 \in R^n$:

$$\lim_{t \to \infty} \mathrm{SUP} \frac{1}{t} \log|X\,(t; t_0, X_0)| < 0$$

Corollary 2. *Assume that there exists a function* $V \in C^{1,2}(R^n \times [t_0, \infty)\,; R_+)$ *and constants* $p > 0$, $c_1 > 0$, $c_2 \in R$, $c_3 \geq 0$ *and* $c_3 > 2c_2$ *such that for all* $X \neq 0$ *and* $t \geq t_0$,

$$(i)\; c_1 |X|^p \leq V\,(X, t)$$

$$(ii)\; \mathscr{L}\,V\,(X, t) \leq c_2 V\,(X, t)$$

$$(iii)\; |V_x\,(X, t)\,g\,(X, t)|^2 \geq c_3 V^2\,(X, t)$$

where the function $g(X, t)$ *is introduced in (2). Then, the stochastic system described by Itô stochastic Equations 7 is almost surely exponentially stable.*

Definition 3. The perturbed system described by Itô stochastic Equations 7 is said to be stochastically boundedly stable if the nominal system will be mean-square stochastically stable and almost all the sample paths of perturbed model will be ultimately bounded by a small bound.

Now we are able to formulate the problem with which we are dealing. For the stochastic model described in (7) a sliding mode controller is to be investigated so that firstly, the sliding dynamic is mean-square stochastically stable; secondly, the state trajectory of system is driven onto the determined sliding surface, and maintain there for all subsequent time.

In this study, we utilized the PI sliding surface defined as follows:

$$s(t) = Gx(t) - \int_0^t G(A + BK)x(s)ds \quad (8)$$

where $G \in R^{m \times n}$ and $K \in R^{m \times n}$ are constant matrices. The matrix K satisfies $\lambda(A + BK) < 0$ and G is chosen so that GB is non-singular. It can be easily proved that the non-singularity of GB can be guaranteed by selecting $G = B^T Y$ where $Y > 0$, since B is deemed to be full column rank.

It is worth nothing that the sliding surface $s(t)$ defined in (8) is well defined for the solution $X(t)$ of the uncertain model (7). In fact, it can be attained from [30] that the solution $X(t)$ of the stochastic system (7) is calculated as follows:

$$X(t) = X_0 + \int_0^t \left[(AX(s) + BU_a(s)) + f(s) \right] ds + \int_0^t DX(s)dw_1 + \int_0^t EX(s)dw_2 + \int_0^t FX(s)dw_3 , \quad (9)$$

where the last three terms in (9) are the Itô stochastic integrals. (9) implies that $X(t)$ is semi-martingale, since $X(t)$ is composed of finite variation terms and a martingale term [29]. Furthermore, it can be obtained from (8) and (9) such that

$$s(t) = GX_0 - \int_0^t [(GBKX(s) - GBU_a(s)) - Gf(s)]\, ds + \int_0^t GDX(s)dw_1 + \int_0^t GEX(s)dw_2 + \int_0^t GFX(s)dw_3 \tag{10}$$

which implies that $s(t)$ is also an Itô stochastic process. This arises from the fact that in this study, the restrictive condition $GD = GE = GF = 0$ is not imposed on the proposed switching surface. Therefore, the sliding surface $s(t) = 0$ is not a semi-martingale and its time-derivative cannot be calculated. However, considering (10) and by applying infinitesimal generator $\mathscr{L}(.)$ described by (1), one can attain $ds(t)$ as follows:

$$\begin{aligned} ds(t) &= \mathscr{L}(s(t))dt + GDX(t)dw_1 \\ &\quad + GEX(t)dw_2 + GFX(t)dw_3 \end{aligned}$$

which leads to:

$$\begin{aligned} ds(t) &= \{(-GBKX(t) + GBU_a(t)) + Gf(t)\}\, dt \\ &\quad + GDX(t)dw_1 + GEX(t)dw_2 + GFX(t)dw_3 \end{aligned} \tag{11}$$

Equation 11 represents a non-semi-martingale sliding surface, since it is composed of non-finite variation terms and a martingale term. It is well known that if the system trajectories are able to enter the switching surface, then $s(t) = 0$. Therefore, the control low $U(t)$ which holds the system trajectories on sliding surface can be obtained by letting $E\{ds(t)\} = 0$, i.e.

$$E\{ds(t)\} = -(GBKX(t) - GBU_a(t)) + Gf(t) = 0 \tag{12}$$

If the matrix G is chosen such that GB is non-singular, this yields

$$U(t) = KX(t) - (GB)^{-1}Gf(t) \tag{13}$$

Substituting Equation 13 into system (7) results the dynamic equation of the system on sliding surface or sliding mode dynamics as follows:

$$\begin{aligned} d\vec{X} &= \left\{(A + BK)\vec{X} + \left[I_n - B(GB)^{-1}G\right] f(t)\right\} dt \\ &\quad + \left\{D\vec{X}\right\} d\vec{w}_1 + \left\{E\vec{X}\right\} d\vec{w}_2 + \left\{F\vec{X}\right\} d\vec{w}_3, \end{aligned} \tag{14}$$

Now, we are able to tackle with the first task of sliding mode controller design which is analyzing the robustly stochastic stability of the sliding mode dynamic described by Equation 14, and elicit sufficient conditions by means of stochastic extension of Lyapunov stability theory and Itô lemma.

Theorem 1. *Consider the Itô stochastic system (7) with assumption 1, and sliding surface described by (8). If there exist symmetric matrix $\Upsilon > 0$ satisfying following LMI:*

$$Q = 2\Upsilon(A + BK) + D^T\Upsilon D + E^T\Upsilon E + F^T\Upsilon F < 0 \tag{15}$$

With sliding mode matrix $G = B^T\Upsilon$, and if $\left\{I_n - B(GB)^{-1}G\right\} f(t)$ be Euclidian norm-bounded i.e.

$$\left\| \left\{I_n - B(GB)^{-1}G\right\} f(t) \right\| \le \alpha = \left\| I_n - B(GB)^{-1}G \right\| \beta$$

then the sliding dynamic in Equation 14 is stochastically boundedly stable.

Proof. Let the Lyapunov candidate function for the system is selected as follows

$$V = X^T\Upsilon X \tag{16}$$

Selected Lyapunov function satisfies following inequality:

$$\|V(X,t)\| \ge \lambda_{\min}(\Upsilon)\|X\|^2 > 0$$

By Itô lemma, we attain the following differential as [30]:

$$\begin{aligned} dV(X(t),t) &= \mathscr{L}(V(X(t),t))dt + 2X^T\Upsilon DXdw_1 \\ &\quad + 2X^T\Upsilon EXdw_2 + 2X^T\Upsilon FXdw_3 \end{aligned} \tag{17}$$

where infinitesimal generator is as follows:

$$\begin{aligned} \mathscr{L}(V(X(t),t)) &= 2X^T\Upsilon\left\{(A + BK)X + \left[I_n - B(GB)^{-1}G\right] f\right\} \\ &\quad + tr\left\{[DX \;\; EX \;\; FX] \begin{bmatrix} X^TD^T \\ X^TE^T \\ X^TF^T \end{bmatrix} \Upsilon\right\} \\ &= X^TQX + 2X^T\Upsilon\left[I_n - B(GB)^{-1}G\right] f(t) \end{aligned} \tag{18}$$

And

$$Q = 2\Upsilon(A + BK) + D^T\Upsilon D + E^T\Upsilon E + F^T\Upsilon F$$

In order to prove stability of sliding dynamic, it is enough to show that (18) is negative. By letting $Q < 0$, it can be shown that:

$$\mathscr{L}(V(X(t),t)) \le \lambda_{\min}(Q)\|X\|^2 + 2\alpha\,\|\Upsilon\|\,\|X\| \tag{19}$$

Since $\lambda_{\min}(Q) < 0$, consequently infinitesimal generator $\mathcal{L}(V(X(t),t)) < 0$ for all t and $X \in B^c(\eta)$, where $B^c(\eta)$ is complement of the closed ball $B(\eta)$, centered at $X = 0$ with radius $\eta = 2\alpha \|\Upsilon\| / \lambda_{\min}(Q)$. Thereby, based on the corollary. 1 the nominal system is mean-square stochastically stable and the sliding mode dynamic is stochastically boundedly stable. □

Theorem 2. *For the Itô stochastic system (7) with assumption 1, and sliding surface described by (8), If the condition of theorem 1 is satisfied then the system is almost surely exponentially stable, too.*

Proof. Considering Lyapunov candidate function (14), it is implied:

$$\lambda_{\min}(\Upsilon)\|X\|^2 \leq \|V(X,t)\| \leq \lambda_{\max}(\Upsilon)\|X\|^2$$

From the theorem 1 it is proved that

$$\mathcal{L}(V(X(t),t)) < 0$$

Furthermore, following equation is hold for Lyapunov candidate

$$|V_x(X,t)g(X,t)|^2 = \left|2X^T gX\right|^2 \leq 2\lambda_{\max}(g)\|X\|^4$$

Therefore, by selecting $p = 2$, $c_1 = \lambda_{\min}(\Upsilon)$, $c_2 = 0$ and $c_3 = \lambda_{\max}(g)$ and based on corollary 2 it is proved that the stochastic system is almost surely exponentially stable. □

Remark 1. It is noted that in work of [31] the restrictive condition $GD = GE = GF = 0$ is imposed to sliding mode dynamics only to make the sliding surface (11) semi-martingale. On the other hands, G should be design such that GB is non-singular. Apparently, these conditions cannot be satisfied in wide range of practical models such as suspension system. Here, to tackle with this encumbrance we choose a non-semi-martingale sliding surface such that its expectation is equal to zero.

Now we proceed to the second task of sliding mode which is designing an SMC law for stochastic model (7) such that the reachability of the sliding surface (8) is ensured.

Theorem 3. *Consider the Itô stochastic system (7) with assumption 1, and sliding surface described by (8) with $G = B^T Y$ where Y satisfies the LMI described in (15). Then, it can be shown that the mean-square reachability of the sliding surface (8) is assured by SMC law (20) and condition (21):*

$$U_a(t) = -(GB)^{-1}\left[(GAX(t) + \phi s(t)) + \rho(sgn(s(t))\right] \tag{20}$$

$$\|A + BK\| \ \|X(t)\| \geq \|f(t)\| \tag{21}$$

with $\rho > 0$ a small constant.

Proof. By plugging (20) into (12) it follows that:

$$\begin{aligned} E\{ds(t)\} = -(GA + GBK)X(t) - \phi s(t) \\ - \rho \operatorname{sgn}(s(t)) + Gf(t) \end{aligned} \tag{22}$$

to investigate the reachability, we select the Lyapunov function as $V_2(t) = \frac{1}{2}s^T s$. Therefore, we have:

$$\begin{aligned} \dot{V}(t) &= s^T(t)\dot{s}(t) \\ &= s^T(t)\left[-(GA + GBK)X(t) - \phi s(t)\right. \\ &\quad \left. -\rho \operatorname{sgn}(s(t)) + Gf(t)\right] \\ &\leq -\left[\|\phi\| \, \|s(t)\|^2 + \rho \, \|s(t)\| + \right. \\ &\quad \left.(\|G\| \, \|A + BK\| \, \|X(t)\| - \|G\| \, \|f(t)\|) \, \|s(t)\|\right] \end{aligned}$$

If the hitting condition (21) is satisfied, it follows:

$$\dot{V}(t) \leq -\|\phi\| \, \|s(t)\|^2 - \rho \, \|s(t)\| \leq 0$$

Which implies that the mean-square of trajectories of the Itô stochastic system (7) driven into sliding surface $s(t) = 0$ despite the mismatched uncertainty. □

For further improvement, combination of proposed stochastic SMC and adaptive dynamic surface [32] can be applied which can be carried out as future work.

Remark 2. In this paper we deal with sliding mode control of suspension system subject to stochastic uncertainties. The random nature of parameters causes the states and control signal to become stochastic as well. Thus, the chattering effect in sliding surface and input signal are negligible against the fluctuations of random signals.

4 | NUMERICAL SIMULATION

In order to investigate the effectiveness of stochastic robust controller, the performance of the suspension system using stochastic sliding mode controller has been simulated on computer. In the numerical simulations, we have computed the results by applying the Milstein discretization approximation to the Itô stochastic differential Equation 7. In this simulation the following values are selected: $m_s = 280kg$, $m_u = 55kg$, $C_s = 1000Ns/m$, $K_s = 18800N/m$, $K_u = 190000N/m$, $K = [-15840 - 350129100 - 2175]$, $\rho = 0.01$. Solving the LMI (15) resulted the following stochastic sliding mode design matrices:

$$\Upsilon = 10^8 \begin{bmatrix} 0.0145 & 0.0009 & -0.0027 & 0.0000 \\ 0.0009 & 0.0016 & -0.0425 & 0.0004 \\ -0.0027 & -0.0425 & 1.6708 & -0.0177 \\ 0.0000 & 0.0004 & -0.0177 & -0.0004 \end{bmatrix}$$

$$G = 10^4[0.0301 \ 0.1195 - 4.4719 - 0.0795]^T$$

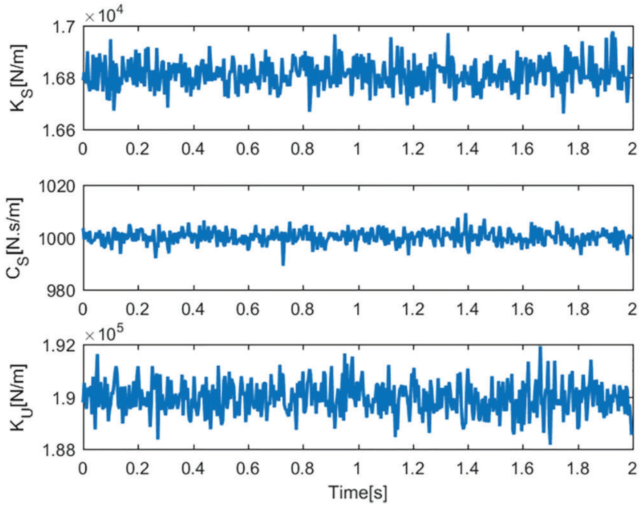

FIGURE 2 Damping and springs characteristics variations [Color figure can be viewed at wileyonlinelibrary.com]

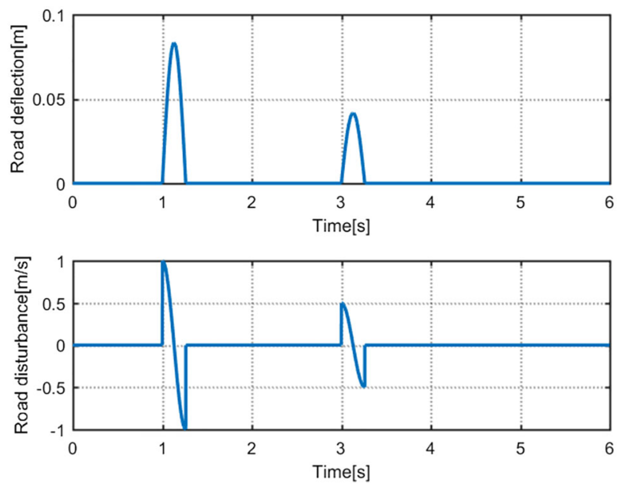

FIGURE 3 Road vertical displacement and velocity (Case 1) [Color figure can be viewed at wileyonlinelibrary.com]

Let us set the variation of system parameters affected by stochastic perturbations as it is depicted in Figure 2.

In order to evaluate the performance of the designed closed-loop active suspension, we consider two typical cases.

4.1 | Case 1

Let the set of road roughness be a sinusoidal bump as it is shown in Figure 3. Here, the mismatched uncertainty is vertical road velocity which conforms a sinusoidal function too, as it is depicted in Figure 3.

Figure 4 shows the performance of perturbed active suspension using stochastic sliding mode controller. As

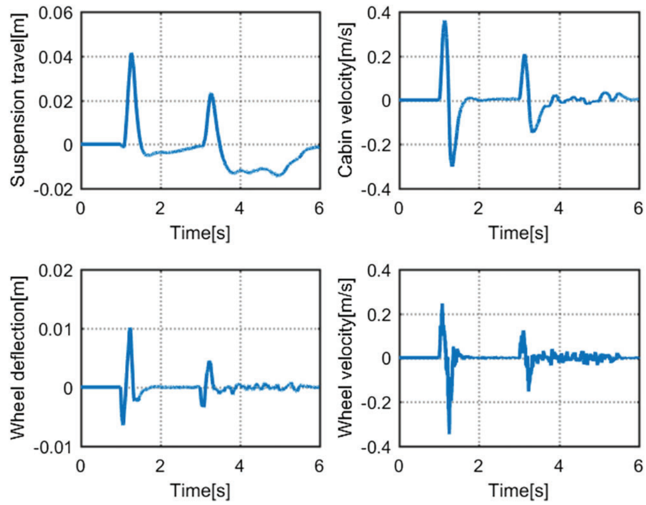

FIGURE 4 State response of active perturbed system (Case 1) [Color figure can be viewed at wileyonlinelibrary.com]

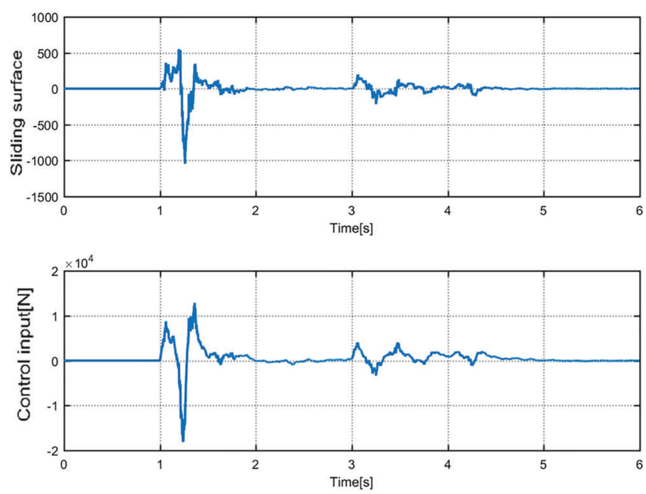

FIGURE 5 Sliding surface and control input (Case 1) [Color figure can be viewed at wileyonlinelibrary.com]

it is clear, the closed loop system stays robust in spite of parametric perturbations and exogenous disturbance. In order to fulfill the objectives of designing an active suspension system, i.e. to provide the ride comfort and road holding, state response of system is observed and simulated. The results show that suspension travel $(x_c - x_w)$ remains within the limited bound $\pm 4\,cm$, and wheel deflection $(x_w - x_r)$ varies within range $\pm 1\,cm$.

Furthermore, the control effort and switching surface are depicted in Figure 5. This illustration shows that the sliding mode is reached after about 4 seconds.

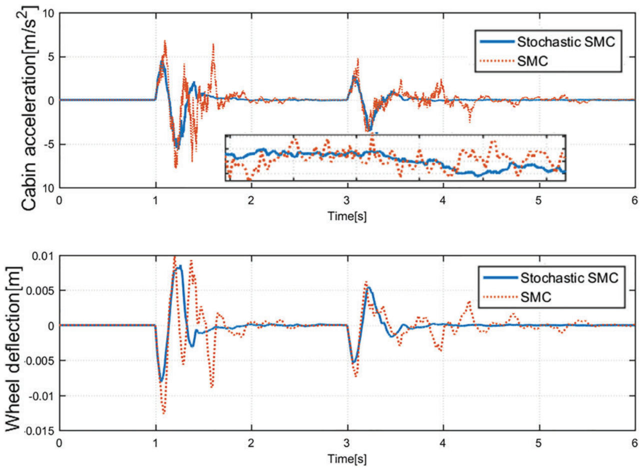

FIGURE 6 Stochastic and non-stochastic SMC responses of perturbed system (Case 1) [Color figure can be viewed at wileyonlinelibrary.com]

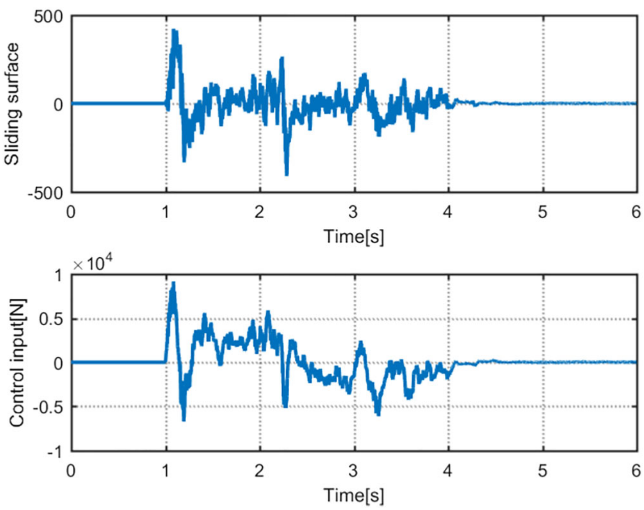

FIGURE 7 Sliding surface and Control input of non-stochastic method (Case 1) [Color figure can be viewed at wileyonlinelibrary.com]

In order to compare the stochastic sliding mode controller with conventional SMC, Figure 6 and Figure 7 illustrate the performance of suspension imposed to parametric perturbations using stochastic and non-stochastic sliding mode controller. Figure 6 indicates that cabin acceleration and wheel deflection of vehicle are more robust against parametric perturbations in the stochastic sliding mode controller.

Based on Figure 6 and Figure 7, the stochastic robust controller outperformed non-stochastic technique in terms of providing ride comfort, road holding, consuming less control effort and also converging faster to the switching surface. Furthermore, non-stochastic controller suffers from more fluctuations in its responses.

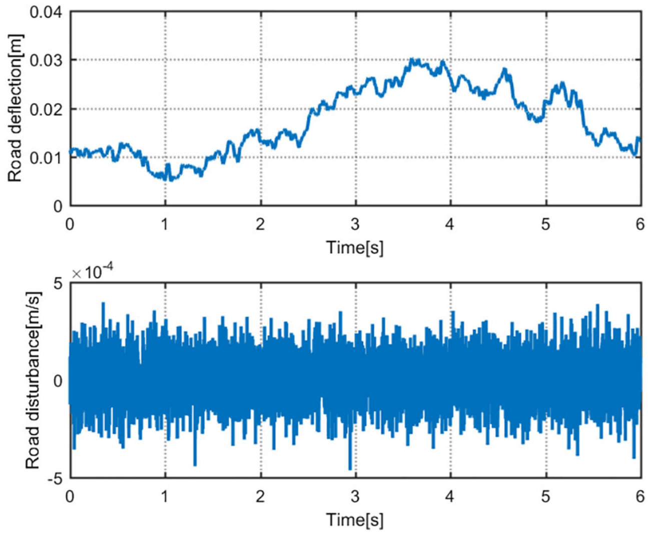

FIGURE 8 Road vertical displacement and velocity (Case 2) [Color figure can be viewed at wileyonlinelibrary.com]

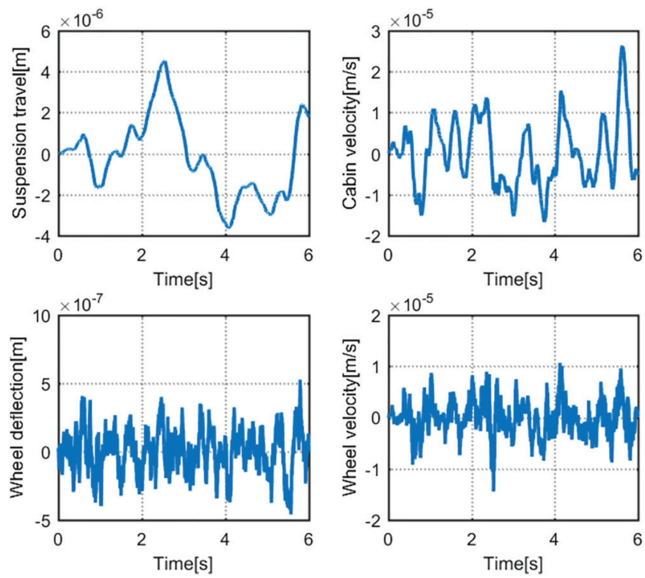

FIGURE 9 State response of active perturbed system (Case 2) [Color figure can be viewed at wileyonlinelibrary.com]

4.2 | Case 2

Here, consider the road surface as a random vibrational profile which is described by the following approximated PSD function:

$$G_q(f) = 4\pi^2 G_q(n_0) {n_0}^2 v,$$

where, $G_q(n_0)$ stands for the road roughness coefficient, n_0 is the reference spatial frequency and v is the vehicle velocity. Selecting $G_q(n_0) = 32 \times 10^{-6} m^3$ and $n_0 = 0.1$ the road profile is depicted in Figure 8. Here, the mismatched uncertainty is vertical road velocity which also is showed in Figure 8.

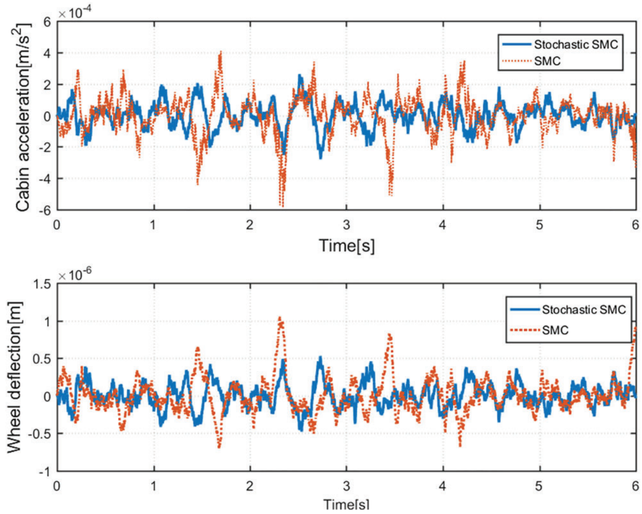

FIGURE 10 Stochastic and non-stochastic SMC responses of perturbed system (Case 2) [Color figure can be viewed at wileyonlinelibrary.com]

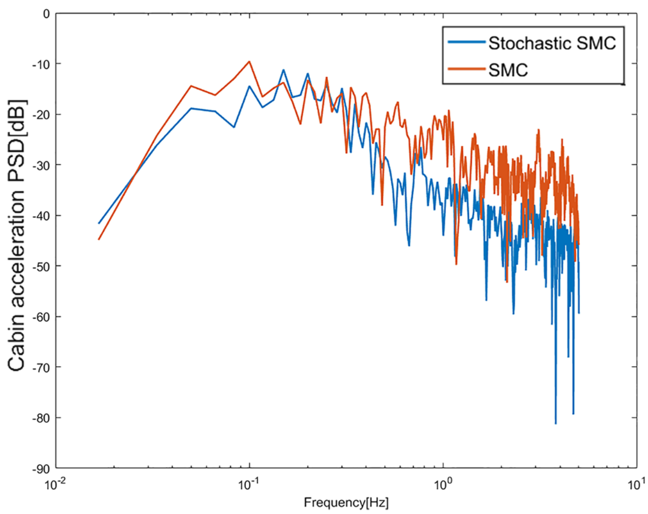

FIGURE 11 Suspension acceleration PSD for Stochastic and non-stochastic SMC (Case 2) [Color figure can be viewed at wileyonlinelibrary.com]

Figure 9 shows the performance of perturbed active suspension using stochastic sliding mode controller. As it is clear, the closed loop system stays robust in spite of parametric perturbations and mismatched disturbance. ride comfort $(x_c - x_w)$, packaging $(x_w - x_r)$ and road holding $\ddot{x}_c$ of suspension is observed and simulated in terms of state responses of system which are restricted within a limited bound.

In order to compare the stochastic sliding mode controller with conventional SMC, Figure 10 illustrates the performance of suspension imposed to parametric perturbations using stochastic and non-stochastic sliding mode controller. Figure 10 indicates that cabin acceleration and wheel deflection of vehicle are more robust against parametric perturbations in the stochastic sliding mode controller.

Figure 11 indicates that the PSD of cabin acceleration is lower in the stochastic-SMC method for the frequency band 1–10 Hz, which is the frequency range closely related to the widely accepted ride comfort level.

5 | CONCLUSIONS

In this paper, the stochastic robust control problem for Itô uncertain model of vehicle suspension have been addressed. The stochastic sliding mode controller is developed in order to cope with mismatched condition and parametric uncertainties of model. The stability analysis is conducted to achieve sufficient condition for stochastic stability of system trajectories on sliding surface. Control law has been synthesized such that the state trajectories of the closed-loop systems are stochastically reached to the specified switching surface. In order to validate performance of control technique, a simulation study is performed for two different cases. The results show that the use of the proposed stochastic sliding mode controller proved to be effective in controlling Itô uncertain model of vehicle suspension and more robust compared to the non-stochastic conventional sliding mode method.

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